ISSN 0104-0499

# PROGRAMA INTERINSTITUCIONAL DE PÓS-GRADUAÇÃO EM ESTATÍSTICA UFSCar-USP

## DEs-UFSCar e SME-ICMC-USP

# THE MIXTURE INVERSE GAUSSIAN DEGRADATION PROCESS IN DEGRADATION DATA AND BURN-IN POLICY

Lia Hanna Martins MORITA
Vera Lucia TOMAZELLA
Narayanaswamy BALAKRISHNAN
Francisco LOUZADA-NETO

## RELATÓRIO TÉCNICO

## TEORIA E MÉTODO – SÉRIE A

Outubro/2016
nº 264

# The Mixture Inverse Gaussian Degradation Process in degradation data and burn-in policy.

Lia Hanna Martins MORITA [1]

Vera Lucia TOMAZELLA [2]

Narayanaswamy BALAKRISHNAN [3]

Francisco LOUZADA-NETO [4]

- ABSTRACT: Analysis of degradation data have become a significant approach to access reliability and security of critical systems. When it is possible to measure degradation, generally we have more information comparing to traditional lifetime data, so that we can evaluate and turn better product reliability. In general, products degrade with age or due to some features called covariates. In addition, degradation is a kind of stochastic process, so it can be modeled in many different approaches. A lot of prediction models were developed for taking into account the concept of degradation. There is a variability of research in the literature about degradation modeling and reliability evaluation using degradation data, see Ma (2007); Meeker and Escobar (1998); Singpurwalla (1995); Van Noortwijk (2009). Degradation is modeled by a stochastic process $Y(t)$, with some proprierties, depending on phenomenons considered, such as Lévy process, Wiener process, Gamma Process and Inverse Gaussian Process. An Inverse Gaussian process allows nonconstant variance and nonzero correlation among data collected at different time points, and the mean function characterizes a monotonic increasing process. In this work, we propose a decision rule for classifying a unit as normal or weak, and give an economic model for determining the optimal termination time and other parameters of a burn-in test. We studied a real laser data set in the literature and developed a simulation study to illustrate the proposed methodology.

- **Key Words**: burn-in test; degradation data; optimal termination time; Inverse Gaussian Process.

# 1 Introduction

As pointed by Meeker and Escobar (1998), some life tests result in a few or no failures. In such cases, it is difficult or impossible to assess reliability with traditional life tests that record only time-to-failure. In this sense, degradation tests are useful to access reliability

[1]Departamento de Estatística, Universidade Federal de Mato Grosso - UFMT, Cuiabá - MT
[2]Departamento de Estatística, Universidade Federal de São Carlos - UFSCar, São Carlos - SP
[3]Statistics, McMaster University
[4]Instituto de Ciências Matemáticas e Computação, Universidade de São Paulo - USP

information of products. Degradation phenomenon is a kind of stochastic process; therefore, it could be modeled in several approaches. Gorjian et al. (2010) presented a review of degradation models, describing a lack of models with advantages and limitations. We can find two techniques of degradation modeling: the general path model introduced by Lu and Meeker (1993); Meeker and Escobar (1998) and models derived from physical principles via stochastic processes. The general path model is the classical approach, which fits the degradation observations by a regression model with random coefficients. The second approach for modeling degradation is through stochastic process. Wang (2010) presented a Wiener process with random effects for degradation data. Tradicional Burn-in policies consist in observe products during a fixed period of time, observing failures, and separating those items before they are shipped to the costumers. However, when we lead with high reliable products, with a few or no failures, we observe the degradation of some quality characteristic, then we construct a cost model to determine the optimal burn-in time. As an example, we consider the light intensity of a light emitting diode (LED), where the light intensity decreases (degrades) over time, and the lamp is considered failed when its intensity reaches a critical value $d$. So, if we can model the degradation path of light intensity properly and classify a unit as normal or weak after observing its initial degradation path. Tsai et al. (2011) used optimal burn-in tests with mixed Gamma Process to describe degradation paths. Tseng et al. (2003) proposed a more flexible burn-in procedure, taking into account the burn-in time and the number of collecting points in the decision rule (called window size), and used Wiener Process to describe the degradation paths. In this work, we use Inverse Gaussian Process (IG Process) to describe the degradation paths.

This paper is organized as follows. Section 2 describes the proposed method, based on Mixture Inverse Gaussian process. In Section 3 we describe the optimal burn-in time procedure, based on the proposed cost model. Section 4 describes the estimation of parameters. Section 5 contains a numerical example on LED, while Section 6 presents a simulation study and Section 7 ends with some comments and conclusion.

## 2 Degradation Model

Let $D(t), t \geq 0$, denote the degradation path of a specific quality characteristic of a product, and $d$ denote its critical level. Then, the product's lifetime is suitably defined as the first passage time when $D(t)$ falls below the critical value $d$. That is,

$$T = inf\left\{t \geq 0 | D(t) \leq d\right\}. \tag{1}$$

the first time when $D(t)$ crosses the critical value $d$ is the product lifetime T.

We propose to use an Inverse Gaussian Process to model the degradation path $D(t)$. This process has monotone paths and was first proposed by Wasan (1968), based on Inverse Gaussian (IG) distribution.

The IG distribution (Chhikara, 1988) has the following p.d.f.

$$f_{IG}(y|\mu,\lambda) = \sqrt{\frac{\lambda}{2\pi y^3}} \times \exp\left[-\frac{\lambda(y-\mu)^2}{2\mu^2 y}\right], \tag{2}$$

where $y > 0$, $\mu > 0$ is the mean and $\lambda > 0$ is the shape parameter.

The c.d.f. for IG distribution is given by

$$F_{IG}(y|\mu,\lambda) = \Phi\left[\frac{\lambda}{y}\left(\frac{y}{\mu}-1\right)\right] + \exp\left(\frac{2\lambda}{\mu}\right)\Phi\left[-\frac{\lambda}{y}\left(\frac{y}{\mu}+1\right)\right] \tag{3}$$

where $\Phi(.)$ is the standard normal c.d.f..

This distribution has close relation with Wiener process with drift, being the first passage time of a Wiener Process with drift.

The degradation path $D(t)$ with IG distribution is given by

$$D(t) \sim IG\left(g(t), \eta g(t)^2\right), \tag{4}$$

where $g(t) > 0, \forall t \geq 0$ is a monotone increasing function, as for example, $g(t) = \mu t$ describes a linear increasing function without intercept, with $\mu > 0$, and $\eta > 0$

The stochastic process in (4) has the following properties:

1. $D(0) = 0$ q.c.;
2. Let $Y = D(t) - D(s)$ the degradation increment in the time interval $[s,t]$, with $t > s > 0$. Then $Y$ has the following distribution

$$Y \sim IG\left(g(t) - g(s), \eta\left(g(t) - g(s)\right)^2\right), \forall t > s \geq 0, \tag{5}$$

   where $g(t) - g(s)$ is the increment in the mean function $g(.)$ in the time interval $[s,t]$.
3. Increments are independent.

The function $g(t)$ has a meaningful interpretation, being the mean function of the process, and the parameter $\eta$ is inversely proportional to the volatility (or variance) of the stochastic process:

$$E\left[D(t)\right] = g(t) \text{ and } VAR\left[D(t)\right] = \frac{g(t)}{\eta}.$$

Suppose there exists a proportion of weak equipments $0 < p < 1$ before conducting a burn-in test. Then a Mixture Inverse Gaussian Model is more appropriated for this kind of data.

Let $g_1(t)$ and $g_2(t)$ the mean functions for the weak group and normal group, respectively, then the degradation paths $D(t)$ are given by the following form:

$$D(t) \sim \begin{cases} IG\left(g_1(t), \eta_1 \left[g_1(t)\right]^2\right), & \text{for weak group,} \\ IG\left(g_2(t), \eta_2 \left[g_2(t)\right]^2\right), & \text{for typical group,} \end{cases} \tag{6}$$

where $g_1(t) > g_2(t) > 0, \forall t \geq 0$, and $\eta_1 > 0$ and $\eta_2 > 0$, that is, the mean function of the degradation process is greater in weak group than normal group.

From (6), we can derive two particular cases:

Case 1: When the degradation paths of typical group have the same volatility parameter $\eta_1 = \eta_2 = \eta$, we have

$$D(t) \sim \begin{cases} IG\left(g_1(t), \eta \left[g_1(t)\right]^2\right), & \text{for weak group,} \\ IG\left(g_2(t), \eta \left[g_2(t)\right]^2\right), & \text{for typical group,} \end{cases} \tag{7}$$

Case 2: Furthermore, when $g_1(t) = g_2(t) = g(t)$, i.e. the degradation paths of typical and weak groups have the same mean function, we have

$$D(t) \sim \begin{cases} IG\left(g(t), \eta_1 \left[g(t)\right]^2\right), & \text{for weak group,} \\ IG\left(g(t), \eta_2 \left[g(t)\right]^2\right), & \text{for typical group,} \end{cases} \tag{8}$$

# 3 Burn-in test and Optimal Burn-in time

A Burn-in test is the process by which components of a system are exercised prior to being placed in service. In general, this process will force certain failures to occur under supervised conditions so an understanding of load capacity of the product can be established. For highly reliable products such as LED lamps we have a few or no failures, then we must propose a method to screen (or classify) the weak items from typical items, based on degradation characteristics of such components. Then we have interest in determine the optimal burn-in time (or termination time) enough to screen the components.

Let $t_b$ the burn-in time. The decision rule used to classify weak units and normal units is given by

R: An item is classified as a normal item at burn-in time $t_b$ if

$$(t_b) \leq \xi(t_b), \tag{9}$$

where $\xi$ denotes the unknown cutoff point at burn-in time $t_b$, which has to be determined.

For fixed burn-in time $t_b$ , we have the following associated misclassification probabilities:

$$\begin{aligned} \alpha(t_b) &= P(\text{misclassifying a normal item as weak item}) \\ &= P(D(t_b) > \xi(t), \\ &= 1 - F_{IG}(g_2(t_b), \eta_2 \left[g_2(t_b)\right]), \end{aligned} \tag{10}$$

where $F_{IG}(.)$ is defined in (3).

In the same way,

$$\begin{aligned}\beta(t_b) &= P(\text{misclassifying a weak item as normal item}) \\ &= P(D(t_b) < \xi(t_b), \\ &= F_{IG}(g_2(t_b), \eta_2\,[g_2(t_b)])\end{aligned} \tag{11}$$

## 3.1 Misclassification cost and optimal cutoff point

Let $n$ denote the total number of units subject to a burn-in test, and $p$ denote the proportion of the weak units. Since we have the probabilities of misclassification $\alpha(t_b)$ and $\beta(t_b)$ given in (10) and (11), we can introduce a cost model.

Let $C_\alpha$ the unit cost of misclassifying a typical item as weak item, and $C_\beta$ the unit cost of misclassifying a weak item as typical item, then we have the cost of misclassification the items as a result of the decision rule (9):

$$MC\left(\xi(t_b)\right) = C_\alpha n(1-p)\alpha(t_b) + C_\beta np\beta(t_b), \tag{12}$$

The optimal cutoff points are cutoff points which results in the minimum misclassification cost. The optimal cutoff points $\xi(\hat{t_b})$ are obtained by minimization of (12).

*Theorem 1:* For a fixed burn-in time $t_b$ and under model (6), the optimal cutoff point is obtained by the following equation

$$-\frac{\left(\xi(t_b)-g_1(t_b)\right)^2\eta_1}{2\eta_1}+\frac{\left(\xi(t_b)-g_2(t_b)\right)^2\eta_2}{2\eta_2} = \log\left[\frac{C_a(1-p)\sqrt{\eta_2}g_2(t_b)}{C_b p\sqrt{\eta_1}g_1(t_b)}\right], \tag{13}$$

which has two real roots:

$$\begin{aligned}\widehat{\xi(t_b)_1} &= \frac{g_1(t_b)\eta_1 - g_2(t_b)\eta_2 - \log\left[\frac{C_a(1-p)\sqrt{\eta_2}g_2(t_b)}{C_b p\sqrt{\eta_1}g_1(t_b)}\right]+\frac{1}{2}\sqrt{\Delta}}{\eta_1-\eta_2} \\ \widehat{\xi(t_b)_2} &= \frac{g_1(t_b)\eta_1 - g_2(t_b)\eta_2 - \log\left[\frac{C_a(1-p)\sqrt{\eta_2}g_2(t_b)}{C_b p\sqrt{\eta_1}g_1(t_b)}\right]-\frac{1}{2}\sqrt{\Delta}}{\eta_1-\eta_2},\end{aligned} \tag{14}$$

where $\Delta = -4(\eta_1-\eta_2)(g_1^2(t_b)\eta_1 - g_2^2(t_b)\eta_2 + 4\left(-g_1(t_b)\eta_1 + g_2(t_b)\eta_2 + \log\left[\frac{C_a(1-p)\sqrt{\eta_2}g_2(t_b)}{C_b p\sqrt{\eta_1}g_1(t_b)}\right]\right)^2$

*Proof:* The proof is presented in the Appendix.

There are two real roots for (13), and we have to check the value of second derivative with respect to $\xi(t_b)$.

The second derivative of (12) with respect to $\xi(t_b)$ is given by

$$\frac{\partial^2 MC\left(\xi(t_b)\right)}{\partial^2 \xi(t_b)} = \frac{n\left(C_b e^{-\frac{\left(\xi(t_b)-g_1(t_b)\right)^2\eta_1}{2\xi(t_b)}} g_1(t_b) p \sqrt{\frac{\eta_1}{\xi(t_b)}} a_1 - C_a e^{-\frac{\left(\xi(t_b)-g_2(t_b)\right)^2\eta_2}{2\xi(t_b)}} g_2(t_b)(-1+p)\sqrt{\frac{\eta_2}{\xi(t_b)}} a_2\right)}{2\xi(t_b)^3\sqrt{2\pi}}, \tag{15}$$

where $a_1 = -3\xi(t_b) - \xi(t_b)^2\eta_1 + g_1(t_b)^2\eta_1$ and $a_2 = 3\xi(t_b) + \xi(t_b)^2\eta_2 - g_2(t_b)^2\eta_2$.

We must take the solution which results in function (15) being positive, which means we found out a global minimum value for the misclassification cost in (12).

*Corollary 1:* For a fixed burn-in time $t_b$ and under model (7), the optimal cutoff point is given by

$$\widehat{\xi(t_b)} = \frac{\left(g_1(t_b) - g_2(t_b)\right)\left(g_1(t_b) + g_2(t_b)\right)\eta}{\left(2\left(g_1(t_b) - g_2(t_b)\right)\eta - 2\log\left[\frac{C_a(1-p)g_2(t_b)}{C_b p g_1(t_b)}\right]\right.} \tag{16}$$

*Proof:* The proof is presented in the Appendix.

*Corollary 2:* For a fixed burn-in time $t_b$ and under model (8), the optimal cutoff point is obtained by the following equation

$$-\frac{\left(\xi(t_b) - g(t_b)\right)^2\eta_1}{2\eta_1} + \frac{\left(\xi(t_b) - g(t_b)\right)^2\eta_2}{2\eta_2} = \log\left[\frac{C_a(1-p)\sqrt{\eta_2}}{C_b p\sqrt{\eta_1}}\right], \tag{17}$$

which has two real roots:

$$\begin{aligned} \widehat{\xi(t_b)_1} &= \frac{g(t_b)\eta_1 - g(t_b)\eta_2 - \log\left[\frac{C_a(1-p)\sqrt{\eta_2}}{C_b p\sqrt{\eta_1}}\right] + \sqrt{\Delta}}{\eta_1 - \eta_2} \\ \widehat{\xi(t_b)_2} &= \frac{g(t_b)\eta_1 - g(t_b)\eta_2 - \log\left[\frac{C_a(1-p)\sqrt{\eta_2}}{C_b p\sqrt{\eta_1}}\right] - \sqrt{\Delta}}{\eta_1 - \eta_2}, \end{aligned} \tag{18}$$

where $\Delta = -4g^2(t_b)(\eta_1 - \eta_2)^2 + \left(2g(t_b)(\eta_1 - \eta_2) - 2\log\left[\frac{C_a(1-p)\sqrt{\eta_2}}{C_b p\sqrt{\eta_1}}\right]\right)^2$.

*Proof:* The proof is presented in the Appendix.

In addiction to the misclassification cost, we also need to pay attention to test costs that include the cost of conducting the degradation test, and the cost of measuring the data.

Suppose $t = 0, t_1, \ldots, t_l$ are the check points of a burn-in test, then the total number of data collection points at $t_b$ is $b+1$ for $1 \leq b \leq l$.

Let $C_{op}$ the cost of operating the degradation test per unit of time, such as labor and indirect labor costs and $C_{mea}$ the cost of measuring data on a unit, such as costs of setting up the measuring equipment.

The total cost of misclassification for each burn-in time $t_b$ is given by

$$TC\left(\xi(t_b)\right) = MC\left(\xi(t_b)\right) + C_{op} \times n \times t_b + C_{mea} \times n \times (b+1). \tag{19}$$

where $MC\left(\xi(t_b)\right)$ is given in (12). The optimal burn-in time $t_b$ is obtained by minimizing (19).

# 4 Likelihood function

Consider a sample of $n$ units, being observed up to time $t_b$ (burn-in time), with collecting points $t_0, t_1, t_2, \ldots, t_b$ and degradation values $D(t_1), \ldots, D(t_b)$.

For each unit $i$ and $1 \leq j \leq b$, define $Y_{ij} = D(t_j) - D(t_{j-1})$ the degradation increment in the time interval $[t_{j-1}, t_j]$. Considering (5) and model (6), the probability density function for $Y_{ij}$ is given by:

$$f_{Y_{ij}}(y_{ij}) = p\sqrt{\frac{\eta_1}{2\pi y_{ij}^3}}\Delta g_1(t_j)e^{-\frac{\eta_1(y_{ij}-\Delta g_1(t_j))^2}{2y_{ij}}} + (1-p)\sqrt{\frac{\eta_2}{2\pi y_{ij}^3}}\Delta g_2(t_j)e^{-\frac{\eta_2(y_{ij}-\Delta g_2(t_j))^2}{2y_{ij}}} \tag{20}$$

where $\Delta g_1(t_j) = g_1(t_j) - g_1(t_{j-1})$ is the time-function increment in the time interval $[t_{j-1}, t_j]$ under weak items and $\Delta g_2(t_j) = g_2(t_j) - g_2(t_{j-1})$ is the time-function increment the time interval $[t_{j-1}, t_j]$ under normal items.

The likelihood function is given by:

$$\begin{aligned} L(g_1(t)), g_2(t), \eta_1, \eta_2, p) &= \prod_{i=1}^{n}\left[p\prod_{j=1}^{b}\sqrt{\frac{\eta_1}{2\pi y_{ij}^3}}\Delta g_1(t_j)e^{-\frac{\eta_1(y_{ij}-\Delta g_1(t_j))^2}{2y_{ij}}} + \right. \\ &\left. (1-p)\prod_{j=1}^{b}\sqrt{\frac{\eta_2}{2\pi y_{ij}^3}}\Delta g_2(t_j)e^{-\frac{\eta_2(y_{ij}-\Delta g_2(t_j))^2}{2y_{ij}}}\right]. \end{aligned} \tag{21}$$

And the log-likelihood function is given by:

$$\begin{aligned} l(g_1(t), g_2(t), \eta_1, \eta_2, p) &= \sum_{i=1}^{n}\left[\log p\prod_{j=1}^{b}\sqrt{\frac{\eta_1}{2\pi y_{ij}^3}}\Delta g_1(t_j)e^{-\frac{\eta_1(y_{ij}-\Delta g_1(t_j))^2}{2y_{ij}}}\right. \\ &\left. + (1-p)\prod_{j=1}^{b}\sqrt{\frac{\eta_2}{2\pi y_{ij}^3}}\Delta g_2(t_j)e^{-\frac{\eta_2(y_{ij}-\Delta g_2(t_j))^2}{2y_{ij}}}\right] \end{aligned} \tag{22}$$

The functions $g_1(.)$ and $g_2(.)$ have to be specified, being indexed by a parameter vector. The maximum likelihood estimates (MLEs) for the parameters can be obtained by direct maximization of (22) with respect to the parameters. Intervals estimates and hypothesis tests are obtained asymptotically.

Similarly, the likelihood functions for model (7) and (8) are obtained.

The criterion used to evaluate the different models is the Akaike information criterion (AIC), which is frequently used in statistical literature in model selection and defined as

$$AIC = 2m - 2l, \tag{23}$$

where $m$ is the number of model parameters, and $l$ is the maximized value of the log-likelihood function of the estimated model. The model with the smallest AIC among all models is selected as the best fitting model.

# 5 Application with laser data

Some devices for light amplification by the stimulated emission of called LASER (Light Amplication by Stimulated Emission of Radiation) present degradation over time, which brings a reduction of the emitted light. This luminosity can be maintained substantially constant, with an increase of operating current. When this current reaches a very high value, it is considered that there was a device failure. Meeker and Escobar (1998) presented a study with degradation data of 15 LASER units from GaAs type (compound with Gallium and Arsenic elements), with observations made at 4.000 hours of operation, with time intervals: $t_0 = 0, t_1 = 250, t_2 = 500, t_3 = 750, \ldots, t_{16} = 4.000$. For each unit and time, the degradation measure is the percent increase in current over time, related to the nominal current. In this experiment the degradation measures are the percentage of increase in current for each unit, and laser unit is considered not working when degradation measure reachs 10% (threshold=10%). Figure 1 shows the degradation paths for Laser data.

From Figure 1 we conclude that the Mixture IG degradation Process can be represented by two mean functions $g_1(t) = \mu_1 t$ and $g_2(t) = \mu_2 t$, for weak group and typical group, respectively, with $\mu_1 > \mu_2 > 0$.

In this laser data, we treated three items having higher degradation paths as weak group, while twelve items having lower degradation paths are in the typical group.

In addition, we present the Mixture Wiener Process model (Tseng and Tang, 2001) and the Mixture Gamma Process (Tsai et al., 2011) for analysis of laser degradation data. The aim here is to compare Mixture IG Process model with these well known models in the literature.

The Mixture Wiener Process (Tseng and Tang, 2001) for the degradation up to time t $D(t)$ is given by

$$D(t) \sim \begin{cases} \mu_1 t + \sigma B(t), & \text{for weak group,} \\ \mu_2 t + \sigma B(t), & \text{for typical group,} \end{cases} \tag{24}$$

where $\mu_1$ and $\mu_2$ denote the drift parameters for weak and typical groups, respectively, $\sigma$ is diffusion coefficient and $B(t)$ is a standard Brownian motion.

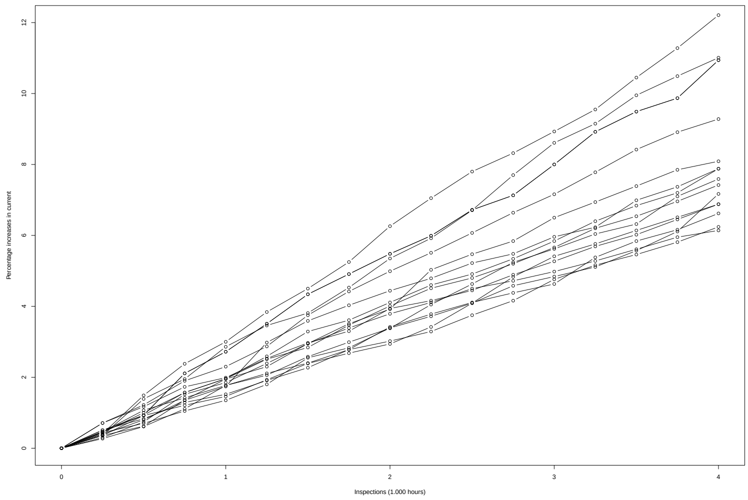


Figure 1: Degradation paths of GaAs laser current.

The Mixture Gamma Process (Tsai et al., 2011) for the degradation up to time t $D(t)$ is given by

$$D(t) \sim \begin{cases} \text{Gamma}\left(g_1(t), \nu\right) & \text{for weak group,} \\ \text{Gamma}\left(g_2(t), \nu\right), & \text{for typical group,} \end{cases} \tag{25}$$

where $g_1(t) > g_2(t)$ are shape functions that take the forms $g_1(t) = \mu_1 t$ and $g_2(t) = \mu_2 t$ with $\mu_1 > \mu_2$ and $\nu > 0$ is the scale parameter.

Figure 2 shows the quantile-quantile (Q-Q) plot for the degradation increments, considering models (6), (7), (8), (24) and (25).

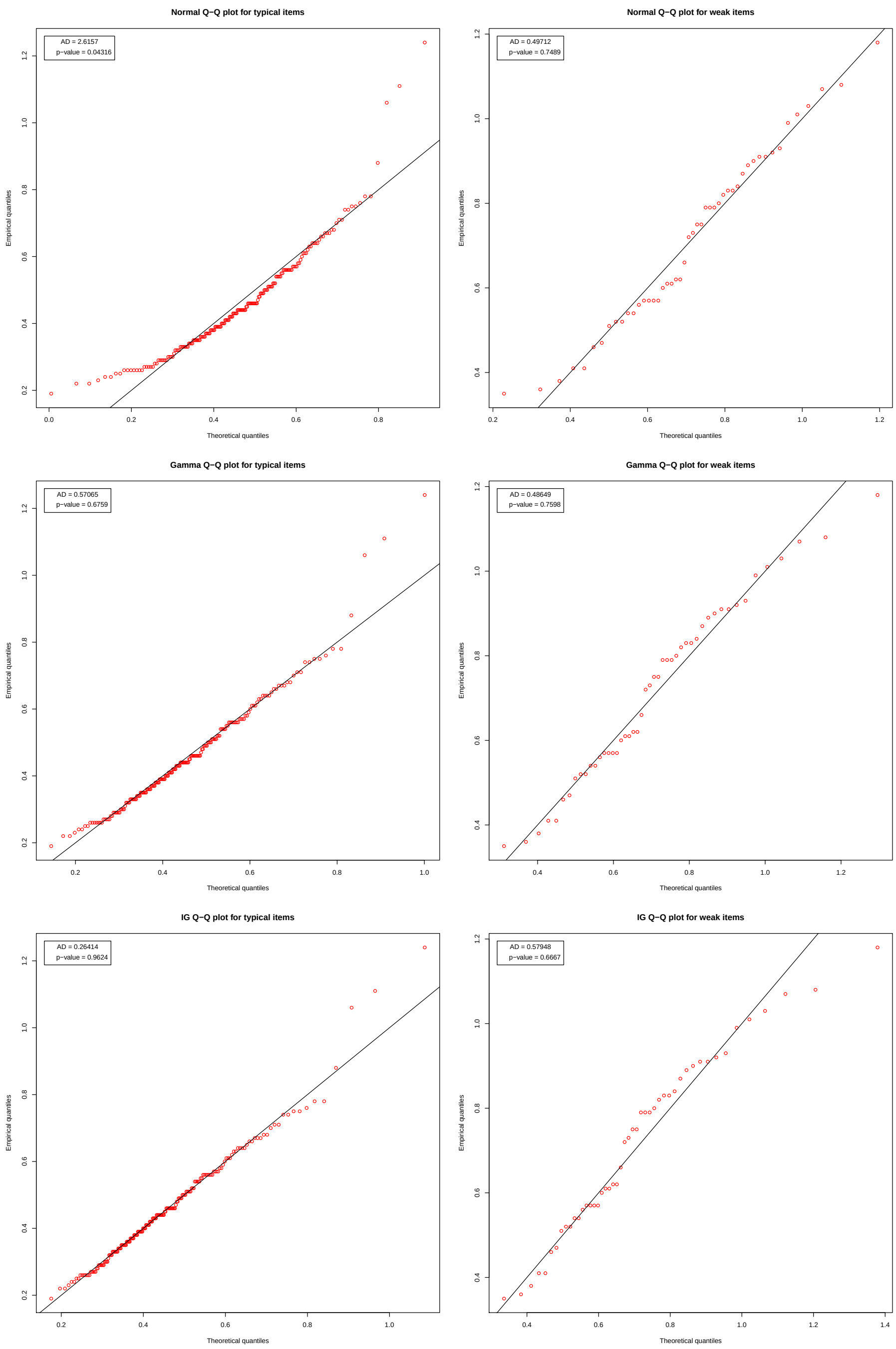


Figure 2: Q-Q plots for laser data.

The results show that Gamma and IG Processes are more suitable for describing laser data than Wiener Process (p-value $< 0.05$ for typical items).

Table 1 show the MLEs for parameters and the corresponding log-likelihood and AIC values for models (6), (7), (8), (24) and (25).

Table 1: MLEs for parameters and AIC values, considering Laser dataset.

| Model | Parameters | MLEs | logL | AIC |
|---|---|---|---|---|
| (6) | $\mu_1$ | 0.0027 | 102.13 | $-194.26$ |
| | $\mu_2$ | 0.0018 | | |
| | $\eta_1$ | 16.0320 | | |
| | $\eta_2$ | 19.2410 | | |
| | $p$ | 0.2660 | | |
| (7) | $\mu_1$ | 0.0027 | 101.79 | $-195.59$ |
| | $\mu_2$ | 0.0018 | | |
| | $\eta$ | 18.234 | | |
| | $p$ | 0.2661 | | |
| (8) | $\mu_1$ | 0.0020 | 76.583 | $-145.17$ |
| | $\eta_1$ | 31.8490 | | |
| | $\eta_2$ | 11.7470 | | |
| | $p$ | 0.1610 | | |
| (24) | $\mu_1$ | 0.0028 | 73.6541 | $-139.31$ |
| | $\mu_2$ | 0.0018 | | |
| | $\sigma$ | 0.0109 | | |
| | $p$ | 0.2156 | | |
| (25) | $\mu_1$ | 0.0519 | 97.1152 | $-186.23$ |
| | $\mu_2$ | 0.0345 | | |
| | $\nu$ | 0.0521 | | |
| | $p$ | 0.2646 | | |

From Table 1, we note that the model (7) has the smallest AIC value. This IG process consider different mean functions for weak group and typical group and equals volatility parameters for both groups. Considering the model (7) which has the best fit in laser degradation data, we selected this model for illustrating the optimal burn-in policy.

Let $C_\alpha = 65$, $C_\beta = 90$, $C_{op} = 0.0009$ and $C_{mea} = 0.0005$ then we can determine the burn in test quantities of interest. Table 2 show the estimated Misclassification Probabilities ((10) and (11)), the Optimal Cutoff point (16) and the Total Cost (19), obtained with the MLEs of parameters.

Table 2: Misclassification Probabilities, Optimal Cutoff point and Total Cost.

| $t_b$ | 250 | 500 | 750 | 1000 | 1250 | 1500 | 1750 | 2000 |
|---|---|---|---|---|---|---|---|---|
| $\xi^*(t_b)$ | 0.60412 | 1.1639 | 1.7247 | 2.2858 | 2.8470 | 3.4082 | 3.9694 | 4.5307 |
| $\alpha(t_b)$ | 0.1519 | 0.1204 | 0.0929 | 0.0718 | 0.0557 | 0.0435 | 0.0341 | 0.0268 |
| $\beta(t_b)$ | 0.4054 | 0.2673 | 0.1911 | 0.1414 | 0.1067 | 0.0815 | 0.0629 | 0.0488 |
| T.Cost($t_b$) | 257.7500 | 188.9700 | 145.2900 | 115.7100 | 95.1340 | 80.7090 | 70.6470 | 63.7670 |
| $t_b$ | 2250 | 2500 | 2750 | **3000** | 3250 | 3500 | 3750 | 4000 |
| $\xi^*(t_b)$ | 5.0920 | 5.6533 | 6.2146 | **6.7759** | 7.3372 | 7.8985 | 8.4598 | 9.0211 |
| $\alpha(t_b)$ | 0.0211 | 0.0167 | 0.0132 | **0.0105** | 0.0083 | 0.0066 | 0.0053 | 0.0042 |
| $\beta(t_b)$ | 0.0381 | 0.0299 | 0.0235 | **0.0186** | 0.0147 | 0.0116 | 0.0092 | 0.0074 |
| T.Cost($t_b$) | 59.2490 | 56.5100 | 55.1230 | **54.7750** | 55.2270 | 56.3020 | 57.8620 | 59.8010 |

From Table 2, we observe that the smallest total cost is 54.7750 at time 3000 hours of operation. Then, the optimal burn-in time is 3000 hours and the optimal cutoff point is 6.7759.

# 6 Simulation Study

A simulation study was developed. N=1.000 datasets with sample size n=200 were generated. We generated the datasets through the IG Process model (7). The true values for parameters are the MLEs in Table 1. We made inferences in parameters considering the IG Process model (7), Wiener Process model (24) and Gamma Process model (25). In other words, we generated data with IG Process and estimated parameters with Wiener Process and Gamma Process, then we have the effect of mispecifying a mixed IG process as a Mixed Wiener Process or Gamma Process.

Table 3 show the MLEs for parameters. We note that the MLEs in the simulated data are close to true parameters in Table 1.

Table 3: MLEs for parameters, considering simulated data.

| Model | Parameters | MLEs |
|---|---|---|
| (7) | $\mu_1$ | 0.0027 |
| | $\mu_2$ | 0.0018 |
| | $\eta$ | 18.238 |
| | $p$ | 0.2665 |
| (24) | $\mu_1$ | 0.0027 |
| | $\mu_2$ | 0.0018 |
| | $\sigma$ | 0.0106 |
| | $p$ | 0.2656 |
| (25) | $\mu_1$ | 0.0527 |
| | $\mu_2$ | 0.0352 |
| | $v$ | 0.0521 |
| | $p$ | 0.2670 |

Let $\widehat{\alpha}_{IG}^{(k)}(t), \widehat{\beta}_{IG}^{(k)}(t), \widehat{\xi}_{IG}^{*(k)}(t), \widehat{TC}_{IG}^{(k)}(t)$ denote the estimated misclassification probabilities, the cutoff point, and the total cost of the $k-$trial under model (7), respectively. The misclassification probabilities, the optimal cutoff point and the total cost under model (7) are then estimated empirically as

$$\begin{aligned}
\overline{\alpha}_{IG}(t_b) &= \frac{1}{N}\sum_{k=1}^{N}\widehat{\alpha}_{IG}^{(k)}(t_b),\\
\overline{\beta}_{IG}(t_b) &= \frac{1}{N}\sum_{k=1}^{N}\widehat{\beta}_{IG}^{(k)}(t_b),\\
\overline{\xi}_{IG}^{*}(t_b) &= \frac{1}{N}\sum_{k=1}^{N}\widehat{\xi}_{IG}^{*(k)}(t_b),\\
\overline{TC}_{IG}(t_b) &= \frac{1}{N}\sum_{k=1}^{N}\widehat{TC}_{IG}^{(k)}(t_b).
\end{aligned}$$

In a similar manner, we estimated the misclassification probabilities, the optimal cutoff point and the total cost under model (24), and denoted them by

$$\overline{\alpha}_W(t_b), \overline{\beta}_W(t_b), \overline{\xi}_W(t_b) \text{ and } \overline{TC}_W(t_b),$$

and the misclassification probabilities, the optimal cutoff point and the total cost under model (25) are given by

$$\overline{\alpha}_G(t_b), \overline{\beta}_G(t_b), \overline{\xi}_G(t_b) \text{ and } \overline{TC}_G(t_b).$$

Tables 4, 4 and 6 show the Estimated Misclassification probabilities, optimal cutoff and Total Cost considering models (7), (24) and (25), respectively.

Table 4: Estimated Misclassification probabilities, optimal cutoff and Total Cost considering model (7).

| $t_b$ | 250 | 500 | 750 | 1000 | 1250 | 1500 | 1750 | 2000 |
|---|---|---|---|---|---|---|---|---|
| $\overline{\xi}_{IG}(t_b)$ | 0.6062 | 1.1667 | 1.7288 | 2.2912 | 2.8537 | 3.4163 | 3.9790 | 4.5416 |
| $\overline{\alpha}_{IG}(t_b)$ | 0.1518 | 0.1192 | 0.0914 | 0.0702 | 0.0543 | 0.0421 | 0.0329 | 0.0257 |
| $\overline{\beta}_{IG}(t_b)$ | 0.4034 | 0.2645 | 0.1881 | 0.1385 | 0.1041 | 0.0792 | 0.0608 | 0.0471 |
| $\overline{TC}_{IG}(t_b)$ | 3384.9 | 2470.8 | 1892.4 | 1502.8 | 1233.5 | 1045.9 | 916.13 | 828.30 |
| $t_b$ | 2250 | 2500 | 2750 | **3000** | 3250 | 3500 | 3750 | 4000 |
| $\overline{\xi}_{IG}(t_b)$ | 5.1043 | 5.6670 | 6.2297 | **6.7924** | 7.3551 | 7.9178 | 8.4805 | 9.0432 |
| $\overline{\alpha}_{IG}(t_b)$ | 0.0202 | 0.0159 | 0.0126 | **0.0099** | 0.0079 | 0.0063 | 0.0050 | 0.0040 |
| $\overline{\beta}_{IG}(t_b)$ | 0.0366 | 0.0286 | 0.0224 | **0.0176** | 0.0139 | 0.0110 | 0.0087 | 0.0069 |
| $\overline{TC}_{IG}(t_b)$ | 771.50 | 737.96 | 722.03 | **719.55** | 727.43 | 743.30 | 765.40 | 792.34 |

Table 5: Estimated Misclassification probabilities, optimal cutoff and Total Cost considering Model (24).

| $t_b$ | 250 | 500 | 750 | 1000 | 1250 | 1500 | 1750 | 2000 |
|---|---|---|---|---|---|---|---|---|
| $\overline{\xi}_{W}(t_b)$ | 0.6492 | 1.2122 | 1.7751 | 2.3381 | 2.9011 | 3.4641 | 4.0271 | 4.5900 |
| $\overline{\alpha}_{W}(t_b)$ | 0.1184 | 0.0944 | 0.0718 | 0.0544 | 0.0413 | 0.0315 | 0.0241 | 0.0185 |
| $\overline{\beta}_{W}(t_b)$ | 0.4372 | 0.2784 | 0.1925 | 0.1380 | 0.1011 | 0.0751 | 0.0563 | 0.0425 |
| $\overline{TC}_{W}(t_b)$ | 3233.2 | 2302.5 | 1728.1 | 1351.0 | 1097.3 | 926.14 | 812.15 | 738.91 |
| $t_b$ | 2250 | 2500 | **2750** | 3000 | 3250 | 3500 | 3750 | 4000 |
| $\overline{\xi}_{G}(t_b)$ | 5.1530 | 5.7160 | **6.2790** | 7.4049 | 7.9679 | 8.5309 | 9.0939 | 10.713 |
| $\overline{\alpha}_{G}(t_b)$ | 0.0143 | 0.0110 | **0.0085** | 0.0066 | 0.0051 | 0.0040 | 0.0031 | 0.0024 |
| $\overline{\beta}_{G}(t_b)$ | 0.0323 | 0.247 | **0.0190** | 0.0146 | 0.0113 | 0.0087 | 0.0068 | 0.0053 |
| $\overline{TC}_{G}(t_b)$ | 695.21 | 673.22 | **667.35** | 673.55 | 688.84 | 711.02 | 738.46 | 769.90 |

From Table 4, we observe that the optimal burn-in time considering IG process model is 3000 hours of operation, and the corresponding optimal cutoff point is 6.7924 and the results are similar to Table 2. From Tables 5 and 5, we observe that the optimal burn-in time is 2750 hours for both Wiener and Gamma process models, and the cutoff points are 6.2790 and 6.2077, respectively. Hence, to measure the effect of this model specification, we analyse the relative bias (RB) of type-I and type-II errors of misclassification.

Table 6: Estimated Misclassification probabilities, optimal cutoff and Total Cost considering Model (25).

| $t_b$ | 250 | 500 | 750 | 1000 | 1250 | 1500 | 1750 | 2000 |
|---|---|---|---|---|---|---|---|---|
| $\overline{\xi}^*(t_b)$ | 0.6244 | 1.1813 | 1.7393 | 2.2977 | 2.8562 | 3.4147 | 3.9733 | 4.5318 |
| $\overline{\alpha}(t_b)$ | 0.1300 | 0.1027 | 0.0782 | 0.0595 | 0.0454 | 0.0348 | 0.0268 | 0.0207 |
| $\overline{\beta}(t_b)$ | 0.4264 | 0.2739 | 0.1911 | 0.1382 | 0.1020 | 0.0763 | 0.0577 | 0.0439 |
| $\overline{\text{T.Cost}}(t_b)$ | 3295.2 | 2363.4.15 | 1784.3 | 1401.1 | 1141.1 | 963.76 | 844.21 | 766.02 |
| $t_b$ | 2250 | 2500 | **2750** | 3000 | 3250 | 3500 | 3750 | 4000 |
| $\overline{\xi}(t_b)$ | 5.0904 | 5.6491 | **6.2077** | 6.7663 | 7.3249 | 7.8836 | 8.4422 | 9.0008 |
| $\overline{\alpha}(t_b)$ | 0.0160 | 0.0124 | **0.0097** | 0.0076 | 0.0059 | 0.0046 | 0.0036 | 0.0029 |
| $\overline{\beta}(t_b)$ | 0.0336 | 0.0259 | **0.0200** | 0.0155 | 0.0120 | 0.0094 | 0.0073 | 0.0057 |
| $\overline{\text{T.Cost}}(t_b)$ | 718.00 | 692.28 | **683.24** | 686.74 | 699.77 | 720.06 | 745.90 | 776.02 |

For each burn-in time $t_b$, the RB of model misspecification for a IG degradation model be mistakenly treated as a mixture Wiener process is given by

$$RB_{\alpha_W}(t_b) = \frac{\overline{\alpha}_W(t_b) - \overline{\alpha}_{IG}(t_b)}{\overline{\alpha}_{IG}(t_b)} \text{ and } RB_{\beta_W}(t_b) = \frac{\overline{\beta}_W(t_b) - \overline{\beta}_{IG}(t_b)}{\overline{\beta}_{IG}(t_b)}.$$

Similarly, the RB of model misspecification for a IG degradation model be mistakenly treated as a mixture Gamma process is given by

$$RB_{\alpha_G}(t_b) = \frac{\overline{\alpha}_G(t_b) - \overline{\alpha}_{IG}(t_b)}{\overline{\alpha}_{IG}(t_b)} \text{ and } RB_{\beta_G}(t_b) = \frac{\overline{\beta}_G(t_b) - \overline{\beta}_{IG}(t_b)}{\overline{\beta}_{IG}(t_b)}.$$

Tables 7 and 8 present the relative bias considering Wiener Process model and Gamma process model $RB_{\alpha_W}(t_b)$ and $RB_{\alpha_G}(t_b)$, respectively.

Table 7: Relative Bias of Type I and Type II errors of misclassification.

| $t_b$ | 250 | 500 | 750 | 1000 | 1250 | 1500 | 1750 | 2000 |
|---|---|---|---|---|---|---|---|---|
| $RB_{\alpha_W}(t_b)$ | −0.2200 | −0.2084 | −0.2143 | −0.2252 | −0.2381 | −0.2518 | −0.2657 | −0.2798 |
| $RB_{\beta_W}(t_b)$ | 0.0838 | 0.0524 | 0.0231 | −0.0037 | −0.0286 | −0.0521 | −0.0745 | −0.0959 |
| $t_b$ | 2250 | 2500 | 2750 | 3000 | 3250 | 3500 | 3750 | 4000 |
| $RB_{\alpha_W}(t_b)$ | −0.2938 | −0.3076 | −0.3212 | −0.3346 | −0.3477 | −0.3606 | −0.3732 | −0.3855 |
| $RB_{\beta_W}(t_b)$ | −0.1165 | −0.1363 | −0.1555 | −0.1740 | −0.1919 | −0.2094 | −0.2263 | −0.2427 |

Table 8: Relative Bias of Type I and Type II misclassification probabilities for Gamma Process

| $t_b$ | 250 | 500 | 750 | 1000 | 1250 | 1500 | 1750 | 2000 |
|---|---|---|---|---|---|---|---|---|
| $RB_{\alpha_G}(t_b)$ | $-0.1441$ | $-0.1380$ | $-0.1437$ | $-0.1526$ | $-0.1628$ | $-0.1737$ | $-0.1847$ | $-0.1959$ |
| $RB_{\beta_G}(t_b)$ | 0.0571 | 0.0356 | 0.0156 | $-0.0026$ | $-0.0198$ | $-0.0361$ | $-0.0517$ | $-0.0668$ |
| $t_b$ | 2250 | 2500 | 2750 | 3000 | 3250 | 3500 | 3750 | 4000 |
| $RB_{\alpha_G}(t_b)$ | $-0.2070$ | $-0.2180$ | $-0.2290$ | $-0.2399$ | $-0.2506$ | $-0.2611$ | $-0.2720$ | $-0.2820$ |
| $RB_{\beta_G}(t_b)$ | $-0.0814$ | $-0.0957$ | $-0.1095$ | $-0.1231$ | $-0.1363$ | $-0.1492$ | $-0.1619$ | $-0.1743$ |

From Table 7 we note that type-I error of misclassification is underestimated for all burn-in times, while type-II error of misclassification is overestimated only the first three burn-in times and underestimated for all other burn-in times. Moreover, the most negative value for $RB_{\alpha_G}$ is $-38.55\%$ in burn-in time 4000 hours of operation and the most negative value for $RB_{\beta_G}$ is $-24,27\%$ in burn-in time 4000 hours of operation.

From Table 8 we note that type-I error of misclassification is underestimated for all burn-in times, while type-II error of misclassification is overestimated only the first three burn-in times and underestimated for all other burn-in times. Moreover, the most negative value for $RB_{\alpha_G}$ is $-28.20\%$ in burn-in time 4000 hours of operation and the most negative value for $RB_{\beta_G}$ is $-17,43\%$ in burn-in time 4000 hours of operation.

This results show that model misspecification influences in the missclassification probabilities, which impacts in the optimal burn-in costs.

# 7 Final Remarks

In highly reliable products where we have a few or no failures, is quite dificult to determine the optimal burn-in time before such product be derivered to costumers, then the optimal burn-in time can be obtained with the degradation values. In this paper we proposed a Mixture IG Process to model the degradation paths of the products, and we presented a decision rule in order separate the weak items to typical items. Then we determined the optimal cutoff points and the optimal burn-in time based on a cost model. In the application with real data, the proposed IG process fits better than Mixture Wiener and Gamma process. In the simulated data, we have shown that model misspecification results influences in the missclassification probabilities and impacts in the optimal burn-in costs.

APPENDIX

Proof of Theorem 1:

Taking the first derivative of (12) with respect to $\xi(t_b)$, we have

$$\frac{\partial MC\left(\xi(t_b)\right)}{\partial \xi(t_b)} = \frac{n\left(C_b e^{-\frac{\left(\xi(t_b)-g_1(t_b)\right)^2\eta_1}{2\xi(t_b)}} g_1(t_b)p\sqrt{\frac{\eta_1}{\xi(t_b)}} + C_a e^{-\frac{\left(\xi(t_b)-g_2(t_b)\right)^2\eta_2}{2\xi(t_b)}} g_2(t_b)(-1+p)\sqrt{\frac{\eta_2}{\xi(t_b)}}\right)}{\xi(t_b)\sqrt{2\pi}} \tag{26}$$

Setting (26) in zero

$$\frac{\partial MC\left(\xi(t_b)\right)}{\partial \xi(t_b)} = 0,$$

then we obtain (13). The two roots are obtained through Bhaskara formula.

Proof of Corollary 1:

Assuming $\eta_1 = \eta_2 = \eta$ in (13) then we have a linear equation with respect to $\xi(t_b)$.

Proof of Corollary 2:

Assuming $g_1(t_b) = g_2(t_b) = g(t_b)$ in (13) then we have (17) and the optimal cutoff point is obtained similarly to *Theorem 1*.

# References


Chhikara, R. (1988). *The Inverse Gaussian Distribution: Theory: Methodology, and Applications*, volume 95. CRC Press.

Gorjian, N., Ma, L., Mittinty, M., Yarlagadda, P., and Sun, Y. (2010). A review on degradation models in reliability analysis. In *Engineering Asset Lifecycle Management*, pages 369–384. Springer.

Lu, C. J. and Meeker, W. O. (1993). Using degradation measures to estimate a time-to-failure distribution. *Technometrics*, 35(2):161–174.

Ma, L. (2007). Condition monitoring in engineering asset management. *Asia Pacific Vibration Conference*, pages 6–9.

Meeker, W. Q. and Escobar, L. A. (1998). *Statistical methods for reliability data*, volume 314. John Wiley & Sons.

Singpurwalla, N. D. (1995). Survival in dynamic environments. *Statistical Science*, pages 86–103.

Tsai, C.-C., Tseng, S.-T., and Balakrishnan, N. (2011). Optimal burn-in policy for highly reliable products using gamma degradation process. *Reliability, IEEE Transactions on*, 60(1):234–245.

Tseng, S. and Tang, J. (2001). Optimal burn-in time for highly reliable products. *International Journal of Industrial Engineering-Theory Applications and Practice*, 8(4).

Tseng, S.-T., Tang, J., and Ku, I.-H. (2003). Determination of burn-in parameters and residual life for highly reliable products. *Naval Research Logistics (NRL)*, 50(1):1–14.

Van Noortwijk, J. (2009). A survey of the application of gamma processes in maintenance. *Reliability Engineering & System Safety*, 94(1):2–21.

Wang, X. (2010). Wiener processes with random effects for degradation data. *Journal of Multivariate Analysis*, 101(2):340–351.

Wasan, M. (1968). On an inverse gaussian process. *Scandinavian Actuarial Journal*, 1968(1-2):69–96.