Some limit theorems for a minimal random walk model

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Outline

- Introduction
- The model and its moments
- Anomalous and normal diffusion
- Results: SLLN, CLT, almost sure covergence and more
- Gaussian fluctuation
- Connection to percolation on random recursive trees

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- LIL:

$$\frac{n^{-1}S_n - \mu}{(2n^{-1}\log\log n)^{1/2}} = \zeta_n,$$

where ζ_n has its set of a.s. limit point in $[-1, 1]$ and $\limsup_n |\zeta_n| = 1$ a.s.

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- Is it possible to mimic the CLT and LIL in this case? Specically, is it possible to find $B_n \to \infty$ such that
$$B_n(S - S_n) \xrightarrow{d} N(0,1) \quad ?$$
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- On central limit and iterated logarithm supplements to the martingale convergence theorem, J. App. Prob. 14, 758-775 (1977).

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- The memory consists of the set of random variables $\eta_{n'}$.The walker remembers as follows:

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- Kumar et al. (2014) showed that

$$\mathbb{E}[S_n] = \frac{qn}{1-\alpha} + (s - \frac{q}{1-\alpha}) \frac{\Gamma(n+\alpha)}{\Gamma(1+\alpha)\Gamma(n)}$$

where Γ is the gamma function.

Second moment

- If $q > 0$, then

$$V[S_n^2] \sim f(\alpha, q, s) \begin{cases} n^{2\alpha}, & \text{if} \quad \alpha > 1/2 \\ n \log n, & \text{if} \quad \alpha = 1/2 \\ n & \text{if} \quad \alpha < 1/2 \end{cases} .$$

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- If $q = 0$, then

$$V(S_n^2) \sim f(p, s) n^{2p}.$$

- Therefore, S_n presents the so-called anomalous diffusion.

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- H is called the Hurst exponent. Usually, it is related to long term correlations.
- Note that sums of i.i.d. random variables always exhibit normal diffusion.
- Indeed, if $\{X_i\}_{i\geq 1}$ is a seq. of iid r.v. with $\mathbb{E}(X_i) = \mu$ and $V(X_i) = \sigma^2$, then

$$V(S_n^2) = n\sigma^2.$$

SLLN

- Thm: Let $(S_n)_{n\geq 1}$ be our model. Then

$$\lim_{n\to\infty} \frac{S_n - \mathbb{E}[S_n]}{n} = 0 \quad \text{a.s.}$$

for any value of $\alpha \in [-1, 1)$. In other words,

$$\lim_{n\to\infty} \frac{S_n}{n} = \frac{q}{1-\alpha} \quad \text{a.s.}$$

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- Remark: The case $\alpha = p - q = 1$ is not covered by SLLN. In fact, if $p = 1$ and $q = 0$, the walk is trivial since by definition its dynamics is determined by the first step η_1, that is, $\eta_n = \eta_1$ for all n ≥ 1.

CLT and LIL

- Thm: Consider $\alpha \leq 1/2$ and $q > 0$.

  a) If $\alpha < 1/2$, then

$$\frac{S_n - \frac{q}{1-\alpha} n}{\sqrt{n}} \xrightarrow{d} N\left(0, \frac{q(1-p)}{(1-\alpha)^2(1-2\alpha)}\right).$$

  b) If $\alpha = 1/2$, then

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Almost sure convergence

- Thm: Consider $q = 0$ and $1/2 < p < 1$, then

$$\frac{S_n}{n^p \Gamma(1+p)^{-1}} - s \to M \text{ a.s. and in } L^d \text{ for } d \geq 1,$$

where M is a non-normal random variable such that

$$\mathbb{E}(M) = 0$$
$$\mathbb{E}(M^2) = \frac{2s\Gamma(1+p)^2}{\Gamma(1+2p)} - s^2$$
$$\mathbb{E}(M^3) = \frac{6s\Gamma(1+p)^3}{\Gamma(1+3p)} - \frac{6s^2\Gamma(1+p)^2}{\Gamma(1+2p)} + 2s^3$$
$$\mathbb{E}(M^4) = \frac{24s\Gamma(1+p)^4}{\Gamma(1+4p)} - \frac{24s^2\Gamma(1+p)^3}{\Gamma(1+3p)} + \frac{12s^3\Gamma(1+p)^2}{\Gamma(1+2p)} - 3s^4.$$

First three moments of S_n

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- $$\begin{aligned}\mathbb{E}(S_n^3) &= \frac{6s\Gamma(n+3p)}{\Gamma(n)\Gamma(1+3p)} - \frac{6s\Gamma(n+2p)}{\Gamma(n)\Gamma(1+2p)} + \frac{s\Gamma(n+p)}{\Gamma(n)\Gamma(1+p)}\\ &\sim \frac{6sn^{3p}}{\Gamma(1+3p)}.\end{aligned}$$

4th moment of S_n

- $$\begin{aligned}\mathbb{E}(S_n^4) &= \frac{24s\Gamma(n+4p)}{\Gamma(n)\Gamma(1+4p)} - \frac{36s\Gamma(n+3p)}{\Gamma(n)\Gamma(1+3p)} \\ &+ \frac{14s\Gamma(n+2p)}{\Gamma(n)\Gamma(1+2p)} - \frac{s\Gamma(n+p)}{\Gamma(n)\Gamma(1+p)} \\ &\sim \frac{24sn^{4p}}{\Gamma(1+4p)}.\end{aligned}$$

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- Remark: Guess $\mathbb{E}(S_n^d) \sim \frac{sd!n^{dp}}{\Gamma(1+dp)}$
- We know S_n/n^p converges a.s. and in L^d because ...

A non-negative martingale

- Consider $q = 0$ and $\mathcal{F}_n = \sigma(\eta_1, \ldots, \eta_n)$. Put

$$a_n = \prod_{j=1}^{n-1} \left(1 + \frac{p}{j}\right) = \frac{\Gamma(n+\alpha)}{\Gamma(1+\alpha)\Gamma(n)} \text{ for } n \geq 2.$$

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$$\mathbb{E}(S_{n+1}|\mathcal{F}_n) = S_n + p\frac{S_n}{n} = \left(1 + \frac{p}{n}\right) S_n.$$

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- Easy to see that $M_n = \frac{S_n}{a_n}$ is a martingale such that $\mathbb{E}(M_n) = s$.
- M_n is non-negative! Doob's convergence theorem implies $M_n \to M$ a.s. for $p \in (0, 1)$.

Mittag-Leffler distribution

- A r.v. X is Mittag-Leffler distributed with parameter $p \in [0, 1]$ if

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- Ps: if $p = 1$, then $X \sim Exp(1)$.

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- Thm: Consider $s=1$ and $p \in (0,1)$. Then

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- Cor:

$$X := \frac{M}{\Gamma(1+p)} = \lim_n \frac{S_n}{n^p}$$

has Mittag-Leffler dist. with parameter p.

Gaussian fluctuation

- If $q = 0$, $1/2 < p < 1$, then

$$\frac{M - M_n}{\sqrt{n^p}} \xrightarrow{d} N\left(0, \frac{1}{\Gamma(1+p)}\right) \text{ as } n \to \infty \text{ and}$$

$$\limsup_{n\to\infty} \frac{|M - M_n|}{\sqrt{2\, n^p \log\log n}} = \frac{1}{\Gamma(1+p)^{1/2}} \text{ a.s.,}$$

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- Here s_n^{-1} plays the role of B_n. ;)

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- For each T_i, $i \geq 1$, T_i is obtained from T_{i-1} by adding a new vertex labeled i linked to a chosen at random vertex u_i from T_i.
- Perform Bernoulli bond percolation on T_n: each edge of T_n is independently removed with prob. $1 - p$.

Percolation connection

- Construct a sequence $\{T_i\}$ of recursive trees: the first graph T_1 consists of a single vertex labeled 1.
- For each T_i, $i \geq 1$, T_i is obtained from T_{i-1} by adding a new vertex labeled i linked to a chosen at random vertex u_i from T_i.
- Perform Bernoulli bond percolation on T_n: each edge of T_n is independently removed with prob. $1 - p$.
- Then the size of the cluster $C_{1,n}$ containing the vertex labeled 1 has the same distribution as the position S_n of the r.w.

References

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Thank you for your attention!